New Applications of Quantum Algebraically Integrable Systems in Fluid Dynamics

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Abstract

Quantum algebraically integrable systems are non-trivial generalizations of Laplacian operators to the case of elliptic operators with variable coefficients. We study corresponding extensions of Laplacian growth connected with algebraically integrable systems, describing viscous free-boundary flows in non-homogenous media. We introduce a class of planar flows related with application of Adler-Moser polynomials and also construct solutions for higher-dimensional cases, where the conformal mapping technique is unavailable.

1 Introduction

We start with an inverse problem in fluid dynamics. Before posing the problem, we explain general settings and notations.

Consider a viscous flow or a flow of liquid in a thin (not necessarily planar) layer of non-homogeneous porous medium. The layer can be viewed as a 2-dimensional surface embedded in the 3-dimensional Euclidean space. We let the layer curvature, permeability, porosity and thickness depend on the surface spatial coordinates x, y. We can choose x, y such that locally

$$dl^2 = G(x, y)(dx^2 + dy^2)$$

where dl is the surface length element. The surface area element is $d\Sigma = Gdxdy$, and the volume of the liquid that can be absorbed in the range $x + dx, y + dy$ equals

$$dV = \eta h d\Sigma = \eta h G dxdy,$$

where $\eta = \eta(x, y), h = h(x, y)$ are the medium porosity and the layer thickness, respectively.

In the porous medium, the flow velocity $v = (dx/dt, dy/dt)$ is proportional to the gradient $\nabla = (\partial/\partial x, \partial/\partial y)$ of the pressure P

$$v = -\frac{\kappa}{\sqrt{G}}\nabla P,$$

where $\kappa = \kappa(x, y)$ is the medium permeability.

It is seen from the above that only the two combinations of variable coefficients, namely

$$\eta h G, \quad \frac{\kappa}{\sqrt{G}}$$

enter the flow equation of motion, and it is convenient to absorb h and G into definitions of the other coefficients. Therefore, without loss of generality, we can consider the flow in the plane parametrized by the complex coordinates $z = x + iy, \bar{z} = x - iy$, choosing η and κ to depend on $z, \bar{z}$, while setting remaining coefficients to unity. It is also convenient to redefine pressure $P \to -P$.

Then the liquid volume conservation leads to the continuity equation

$$(\nabla \cdot \eta v) = 0, \tag{1}$$

while the dynamical law of motion rewrites as

$$v = \kappa \nabla P. \tag{2}$$

We consider a situation where the liquid occupies a bounded, simply-connected open region Ω of the plane, whose time evolution $\Omega = \Omega(t)$ is induced by the flow.

At fixed time t, the pressure is constant along the boundary

$$P(\partial\Omega(t)) = P_0(t). \tag{3}$$

Note, that the in the case of the simply-connected domains considered here, the dynamics is independent of $P_0(t)$, and for convenience the latter can be set to zero.

The normal velocity of the boundary v_n and that of the flow coincide at $\partial\Omega$

$$v_n = n \cdot v \quad \text{if} \quad z \in \partial\Omega. \tag{4}$$

The flow is singularity driven. For instance

$$P \to \frac{q(t)}{\kappa(z_1, \bar{z}_1)\eta(z_1, \bar{z}_1)} \log|z - z_1| + \sum_{j=1}^{k} \left(\frac{\mu_j(t)}{(z - z_1)^j} + \frac{\bar{\mu}_j(t)}{(\bar{z} - \bar{z}_1)^j} \right), \quad \text{as} \quad z \to z_1, \tag{5}$$

when $k + 1$ multipole sources are located in vicinity of $z = z_1$ in interior of the fluid domain Ω.

Equations (1) - (5) set a free boundary problem where evolution of the boundary $\partial\Omega(t)$ is completely determined by the initial condition $\partial\Omega(0)$ and dynamics (strengths) $q(t) = \bar{q}(t), \mu_j(t), j = 1, .., k$ of the sources.

It follows from equations (1), (2) and (5) that the pressure satisfies the following linear PDE

$$(\nabla \kappa \eta \nabla) P = 2\pi \hat{q} [\delta(x - x_1)\delta(y - y_1)] \tag{6}$$

where $\hat{q}$ is a differential operator of the order k

$$\hat{q} = q(t) + \sum_{j=1}^{k} (-1)^j \left(q_j(t) \frac{\partial^j}{\partial z^j} + \bar{q}_j(t) \frac{\partial^j}{\partial \bar{z}^j} \right) \tag{7}$$

and δ denotes Dirac delta-function.

Since permeability and porosity are always positive, the operator in the lhs of the last equation is elliptic and this class of free boundary problems is called elliptic growth. For the case of homogenous media

$$\kappa = \text{const}, \quad \eta = \text{const}, \tag{8}$$

the above operator is Laplacian and a subclass of related growth processes is called Laplacian growth or Hele-Shaw problem (see e.g. [11]).

Inverse problem: Consider an inhomogeneous medium with given permeability $\kappa(x)$ and porosity $\eta(x)$. In this problem one wishes to control the boundary dynamics using finite number of closely placed fixed sources/sinks by varying their strengths in time.

The solution to this problem may be provided by finding a number of sources (or sinks) that better approximate the dynamics in quest. The shape of domain is approximated by time dependent polynomial conformal mapping

$$z = F(w,t) = z_1 + r(t)w + \sum_{i=2}^{l} u_i(t)w^i, \quad \bar{r} = r, \tag{9}$$

from the unit circle $|w| = 1$ in the "mathematical" plane w. For the case both permeability and porosity being functions of one space variable the problem can be solved using the following number of sources

$$k + 1 = \left(\deg(S(x)) + \frac{n(n+1)}{2} + 1\right)\deg(F(w)). \tag{10}$$

Here n is a number of Adler-Moser polynomial $p_n(x)$ and $S(x)$ is a polynomials with free coefficients that approximate characteristics of the medium inside Ω:

$$\kappa\eta = \xi(x)^{-2}, \quad \xi(x) = \frac{p_n(x)}{p_{n-1}(x)}, \quad \eta(x) = p_{n-1}(x)S(x) \tag{11}$$

The intrinsic property of Adler-Moser polynomials $p_n(x)$ consist in possessing $n+1$ real free parameters, two of them coming from shift and scaling symmetry and other $n-1$ defining polynomial coefficients. This, together with freedom in definition of $S(x)$, allows approximation the medium characteristics. Increase in the quality of approximation will, in general, result in increase of the number of sources as seen from (10).

Example: Let us consider simplest example with linear porosity $\eta(x) = x$. In the case the first two Adler-Moser polynomials

$$p_1 = x, \quad p_2 = x^3 + t_1$$

can approximate with $\kappa(x)$ close to the following,

$$\kappa = \frac{x}{(x^3 + t_1)^2}$$

Then, according to (10),

$$k + 1 = 4 \deg(z(w)).$$

For instance, 4 sources are needed to sustain a circular form of domain, i.e. $z = z_1 + r(t)w$, for $\kappa(x)$ of the present example.

The above problem and examples served as motivation for the present paper that have emerged in relation with our recent studies in quantum algebraically integrable systems (QAIS) and in elliptic growth.

In the sequel we will establish important connections between the theory of QAIS, integrable elliptic growth and generalized quadrature identities. Later, we will consider higher-dimension cases, when the conformal mappings are inapplicable and the result is uniquely due to advances in the theory of QAIS.

2 Quantum algebraically integrable systems as generalisations of Laplace operators

Let us review some common facts on QAIS. The classical Liouville integrability is equivalent to existence in a Hamiltonian system a maximal set of Poisson commuting invariants: For a given classical integrable system with n degrees of freedom there cannot exist more than n functionally independent integrals in involution.

In the case of quantum integrable system in n dimensions, integrals form a commutative rings of differential operators generated by n independent elements and such a ring is called complete. Algebraic quantum integrability exhibits when there appears an additional, $n+1$th independent generator of the ring of commuting differential operators. Such rings of commuting differential operators are called overcomplete.

The simplest example of nontrivial QAIS comes from the theory of Calogero-Moser operators in one dimension. This example is a quantum system with the Hamiltonian

$$H = \partial_x^2 - \frac{2}{x^2}.$$

For our purposes, it is convenient consider a gauge-equivalent differential operator

$$L_1 = xHx^{-1} = x^2 \partial_x \frac{1}{x^2} \partial_x = \partial_x^2 - \frac{2}{x} \partial_x$$

For this system, there exists another differential operator

$$M_1 = \partial_x^3 - \frac{3}{x^2} \partial_x + \frac{3}{x^2} \partial_x$$

that commutes with L_1 and does not belong to the ring generated by L_1. Thus, in one dimension, we have the overcomplete commutative ring generated by two independent elements L_1 and M_1. It is important to note that this ring is not trivial in a sense that it is not a subring of any ring

of commuting differential operators generated by a single element, since there is no a first-order integral for the above system.

The following example concerns the 2-dimensional case. In $d = 2$ the ring may be formed by the following set of commuting differential operators:

$$L_1 = x^2\nabla\frac{1}{x^2}\nabla = x^2\partial_x\frac{1}{x^2}\partial_x + \partial_y^2, \quad M_1 = \partial_x^3 - \frac{3}{x^2}\partial_x + \frac{3}{x^2}\partial_x, \quad K_1 = \partial_y^2. \tag{12}$$

Let us note, that operator L_1 is not trivial in the sense that it is not equivalent, up to change of dependent and independent variables and multiplication, to any Beltrami-Laplace operator on some 2-dimensional manifold. The operator L_1 describes elliptic growth with $\kappa\eta = 1/x^2$ in (6).

Another important concept of algebraic integrability is the intertwining operator. Consider the differential identity

$$\Delta T = TL \tag{13}$$

that relates the Laplace operator with a differential operator of a second order L. Operator T is called the intertwining operator and (13) is an intertwining identity.

The simplest example of intertwining identity refers to factorization of the one-dimensional Laplace operator ∂_x^2 :

$$\frac{\partial^2}{\partial x^2} = \left(\frac{1}{x}\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} - 1\right).$$

and

$$\left(x\frac{\partial}{\partial x} - 1\right)\partial_x^2 = \left(x\frac{\partial}{\partial x} - 1\right)\left(\frac{1}{x}\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} - 1\right)$$

from which we obtain the simplest intertwining identity

$$T_1\Delta = L_1T_1 \qquad T_1 = x\frac{\partial}{\partial x} - 1 \tag{14}$$

that relates the Laplacian with the Calogero-Moser operator L_1 of (12).

Higher intertwining identities

$$T_n\Delta = L_nT_n$$

involve operators of higher order, like the following examples:

$$T_n = x^n\left(\frac{\partial}{\partial x} - \frac{n}{x}\right)\left(\frac{\partial}{\partial x} - \frac{n-1}{x}\right)...\left(\frac{\partial}{\partial x} - \frac{1}{x}\right), \quad L_n = \quad x^{2n}\nabla\frac{1}{x^{2n}}\nabla$$

The main purpose of the intertwiner is to produce eigenfunctions of the interlaced operators from eigenfunctions of Laplacian:

$$\Delta\phi = \lambda\phi \qquad \Longrightarrow LT[\phi] = \lambda T[\phi]$$

Classification of QAIS is an open problem. Presently, algebraic integrability is found in systems related with finite-reflection Coxeter groups and their special deformations [10, 7, 6]. In two dimensions, most general class of QIAS is related to soliton solutions of the KDV hierarchy [14, 3], while the algebraically integrable systems with coefficients depending on one variable only are related to rational solutions of the KDV hierarchy (Adler-Moser polynomials) [13, 12].

In the following section we review applications of QAIS to generalization of harmonic analysis emerging in free-boundary problems of fluid dynamics.

3 Mean value integral identities, conservation laws and algebraic domains

Let $\phi(z, \bar{z})$ be a time-independent function satisfying

$$\nabla \kappa \eta \nabla \phi = 0, \quad z \in \Omega \tag{15}$$

in whole Ω, including $z = z_1$.

Let us now estimate the time derivatives of the following quantities

$$M[\phi] = \int_{\Omega(t)} \eta \phi dx dy.$$

Considering an infinitesimal variation of the fluid domain $\Omega(t) \to \Omega(t + dt)$, we get

$$\frac{dM[\phi]}{dt} = \oint_{\partial\Omega(t)} v_n \eta \phi dl,$$

where dl is the boundary arc length. From (2), (3), (4) it follows

$$\frac{dM[\phi]}{dt} = \oint_{\partial\Omega(t)} (P \kappa \eta \nabla \phi - \phi \kappa \eta \nabla P) \cdot n dl.$$

Applying the Stokes theorem and remembering that P and ϕ satisfy (**??**), (15), we get

$$\frac{dM[\phi]}{dt} = 2\pi \hat{q}^*[\phi](z_1, \bar{z}_1). \tag{16}$$

Note that mixed derivatives are absent in $\hat{q}$ (c.f. (7)), for by (15), $\frac{\partial^2 \phi}{\partial z \partial \bar{z}}$ is expressed through first derivatives of ϕ. $\bar{q}_j$ is the complex conjugate of q_j, since both $\phi(z, \bar{z})$ and $\bar{\phi}(\bar{z}, z)$ satisfy (15).

It follows that $M[\phi]$ is conserved for any solution of (15), a such that $\hat{q}^*[\phi](z_1, \bar{z}_1) = 0$.

The conservation laws have been first obtained for the homogeneous medium flows in [9], the variable-coefficient generalization seems to be first presented in [8].

The Laplacian growth (8) is the simplest example, where the conservation laws can be written down explicitly [9]. In this example, any (anti)analytic in Ω function satisfies (15)

$$\phi(z, \bar{z}) = f(z) + g(\bar{z}), \quad \text{for} \quad \kappa = 1, \quad \eta = 1, \tag{17}$$

where f, g are (anti)analytic in Ω and the quantities

$$\int_{\Omega(t)} \left(f(z)(z - z_1)^{k+1} + g(\bar{z})(\bar{z} - \bar{z}_1)^{k+1} \right) dx dy$$

are integrals of motion for the free-boundary flows driven by a multipole source of order k located at $z = z_1$ in homogeneous medium.

Returning to the general case, we integrate (16) getting

$$M[\phi](t) = M[\phi](0) + 2\pi\hat{Q}[\phi](z_1, \bar{z}_1),$$

where

$$\hat{Q} = \int_0^t \hat{q}^*(t')dt' = Q + \sum_{j=1}^{k}\left(Q_j \frac{\partial^j}{\partial z^j} + \bar{Q}_j \frac{\partial^j}{\partial \bar{z}^j}\right), \quad . \tag{18}$$

Therefore $M[\phi](t)$, and consequently a form of the domain, does not depend on the history of the sources and is a function of "multipole fluxes"

$$Q = \int_0^t q(t')dt', \quad Q_j = \int_0^t q_j(t')dt', \quad \bar{Q}_j = \int_0^t \bar{q}_j(t')dt', \quad j = 1..k$$

injected by time t.

Now consider the special case when $M[\phi](0) = 0$ that describes the injection of the fluid to an initially empty medium. In such a case

$$\int_\Omega \eta(z, \bar{z})\phi(z, \bar{z})dxdy = 2\pi\hat{Q}[\phi](z_1, \bar{z}_1), \tag{19}$$

Equation (19) is a generalization of mean value or quadrature identities appearing in the theory of harmonic functions [17] to the case of elliptic equations with variable coefficients. The simplest example of a quadrature identity is a mean value theorem for harmonic functions. Special domains for which the quadrature identities hold are called quadrature domains in the theory of the harmonic functions.

Note, that above derivation can be also adapted to higher dimensional case where similar identities hold.

Now, we return to our inverse free-boundary problem stated in the introduction. It translates to the language of quadrature domains as follows: Given a domain approximated by polynomial conformal mapping (9), find κ and η that best approximate given permeability and porosity and for which a quadrature identity holds in the above domain.

Below, we are going to consider the case of stratified media, when permeability and porosity depend on one coordinate.

4 Planar free-boundary flows in stratified media and rational solutions of the KDV hierarchy

Below we consider solution to the inverse problem for stratified medium presented in the introduction.

Theorem 1 *Let $\Omega(t)$ be a domain in the plane given by a polynomial conformal mapping of the unit disc (9). Then this domain can be formed by injection of fluxes Q and $Q_j, j = 1..k$ through a multipole source of order k at $z = z_1$ in a medium with permeability and porosity given by (11). The order of the source is given by (10).*

The proof is based on a constructive procedure of derivation of a linear system of equations for values of fluxes Q, Q_j that is a verification of quadrature identity (19) for all regular in Ω solutions of the elliptic equation

$$\xi(x)^2 \nabla \frac{1}{\xi(x)^2} \nabla \phi = 0 \tag{20}$$

with ξ given by (11) in an algebraic domain defined by mapping (9). The "x-dependent" part of the above elliptic operator $\xi(x)^2 \partial_x \xi(x)^{-2} \partial_x$ is gauge equivalent to a Schrodinger (Lax) operator connected with rational solutions of the KDV hierarchy with potential expressed through Adler-Moser polynomials [12].

The proof follows general argumentation that was used for system with finite reflection invariance in [1]: Here we just outline a procedure needed for finding values of fluxes.

In the case of equation (20), the intertwiner between the elliptic operator in (20) and Laplacian (cf (13)) can be expressed in a form of a Wronskian [12, 13]

$$T[f](x,y) = \frac{W[\psi_1(x), \dots, \psi_n(x), f(x,y)]}{p_{n-1}(x)} \tag{21}$$

where the Adler-Moser polynomials p_n are also expressed as Wronskians

$$p_n(x) = W[\psi_1(x), \dots, \psi_n(x)]$$

of functions

$$\psi_{i+1}(x)'' = \psi_i(x), \quad \psi_1(x) = x$$

The nth Adler-Moser polynomial depends non-trivially on $n-1$ free parameters ("KDV times") that emerge as integration constants for ψ_ith in the above equation.

It then follows that general solution of (20) can be written down as

$$\phi(x,y) = T[f(z) + g(\bar{z})]$$

with T given by (21) and $f(z), g(\bar{z})$ being functions (anti)analytic in Ω. Taking (11) into account we get

$$\eta\phi = S(x) W[\psi_1(x), \dots, \psi_n(x), f(z) + g(\bar{z})]$$

which is linear differential operator with polynomial in $z, \bar{z}$ coefficients acting on $f(z)$ or $g(\bar{z})$. It is sufficient to consider its action on $f(z)$:

$$\eta\phi = \sum_{i=0}^{n} P_j(z,\bar{z}) \frac{d^j f(z)}{dz^j}.$$

We can now evaluate lhs of the quadrature identity (19). Using Green theorem, we get

$$\int_\Omega \eta\phi dxdy = \sum_{j=0}^{n} \oint_{\partial\Omega} R_j(z,\bar{z}) \frac{d^j f(z)}{dz^j} dz, \quad \frac{\partial R_j(z,\bar{z})}{\partial \bar{z}} = \frac{1}{2i} P_j(z,\bar{z}) \tag{22}$$

where

$$R_j = c_j \bar{z}^{1+\deg(S)+j(j+1)/2} + \text{lower order terms in} \quad \bar{z}, z \tag{23}$$

is a polynomial of $\deg(S) + 1 + j(j+1)/2$ th power in $\bar{z}, z$.

Since $\partial\Omega$ is an image of unit circle under the mapping (9), and since on the unit circle

$$\bar{w} = 1/w, \quad F(\bar{w}) = \bar{F}(1/w)$$

we can rewrite (22) as integrals taken around the unit circle

$$\int_\Omega \eta\phi dxdy = \sum_{j=0}^{n} \oint_{|w|=1} F'(w) R_j(F(w), \bar{F}(1/w)) \left(\frac{1}{F'(w)}\partial_w\right)^j [f(F(w))]dw.$$

The map (9) is analytic and polynomial with $F'(w) \neq 0$ on the unit disc, and since $R_j(z, \bar{z})$ are polynomials in $z, \bar{z}$, the above integrals can be evaluated as a sum of a finite number of residues at $w = 0$. Taking (23) and (9) into account, we get

$$\int_\Omega \eta\phi dxdy = \sum_{j=0}^{(\deg(S)+1+n(n+1)/2)\deg F+n-1} C_j \left(\frac{d^j f(F(w))}{dw^j}\right)_{w=0}$$

Now we consider rhs of quadrature identity (19). According to (18) we have

$$\hat{Q}[\phi](z_1, \bar{z}_1) = \hat{Q}[T[f(z)]]_{z=z_1} =$$

$$Q\sum_{j=1}^{n} A_j \left(\frac{d^j f(F(w))}{dw^j}\right)_{w=0} + \sum_{j=1}^{k}\sum_{m=0}^{n}(Q_j B_{jm} + \bar{Q}_j C_{jm}) \left(\frac{d^{j+m} f(F(w))}{dw^{j+m}}\right)_{w=0}.$$

Equating the above expressions for lhs and rhs of the quadrature identity, we get $k+n+1$ equations for $2k+1$ unknowns $Q, Q_j, \bar{Q}_j, j = 1..k$ with k given by (10). The condition that $\bar{Q}_j$ is a complex conjugate of Q_j and Q is real uniquely defines all fluxes.

Concluding consideration of highly viscous planar flows, it is interesting to note that Adler-Polynomials, and rational Baker-Akhieser function of the KDV hierarchy in general, seem to be ubiquitous in integrable problems of two dimensional fluid mechanics: They also emerge on the other extreme, in the theory of inviscid flows that are finite-dimensional reductions of the Euler equations, describing equilibria and uniform motion of vortices in two dimensions [14, 15, 16].

5 Higher-Dimensional Flows

Three-dimensional viscous free-boundary flows are also of practical interest, while in dimensions four and higher the interest is mainly due to generalization of theory of harmonic functions and mean-value quadrature identities.

Since the conformal mapping technique is unavailable here, we propose to write-down multidimensional quadrature identities using methods of construction of fundamental solutions for QAIS [4].

Let us demonstrate application of these methods by a direct approach using example of an expanding sphere $(\zeta_1-\zeta_1')^2+(\zeta_2-\zeta_2')^2+\ldots+(\zeta_d-\zeta_d')^2=r(t)^2$ centered at $\zeta'=(\zeta_1',\ldots,\zeta_d')$ in d-dimensional Euclidian space. The flow is governed by the following elliptic equation

$$\mathcal{L}[P](\zeta)=\sigma_{d-1}\hat{q}[\delta(\zeta_1-\zeta_1')\ldots\delta(\zeta_d-\zeta_d')], \tag{24}$$

where σ_{d-1} is area of $d-1$ dimensional unit sphere and

$$\mathcal{L}=\nabla\frac{1}{\zeta_1^2}\nabla, \quad \eta=1, \quad \nabla:=(\partial/\partial\zeta_1,\ldots\partial/\partial\zeta_d)\,.$$

This is the simplest non-trivial example of flow in a non-homogeneous medium in higher dimensions that is related to QAIS. Similar results can be obtained for more complex cases of QAIS, such as those related to rational or soliton solutions of KDV hierarchy or finite-reflection groups.

Proposition 1 *Let ϕ be any solution of $\nabla\zeta_1^{-2}\nabla\phi=0$ regular in the closure of the ball*

$$\Omega:\ (\zeta_1-\zeta_1')^2+(\zeta_2-\zeta_2')^2+\ldots+(\zeta_d-\zeta_d')^2<r^2.$$

Then

$$\int_\Omega\phi(\zeta)d\zeta_1\ldots d\zeta_d=v_d\left(\phi(\zeta)+\frac{r^2}{(d+2)\zeta_1'}\frac{\partial\phi(\zeta)}{\partial\zeta_1}\right)_{\zeta=\zeta'},$$

where v_d is volume of Ω, and

$$\int_{\partial\Omega}\phi(\zeta)ds=s_{d-1}\left(\phi(\zeta)+\frac{r^2}{\zeta_1'd}\frac{\partial\phi(\zeta)}{\partial\zeta_1}\right)_{\zeta=\zeta'},$$

where integration is taken over surface of $d-1$ dimensional sphere $\partial\Omega$ centered at $\zeta=\zeta'$, and s_{d-1} denotes area of this sphere.

This is an analog of mean value theorem for harmonic functions.

Proof: is by direct solution of the free-boundary problem (24), (3) in d dimensions. Below we present solution for all dimensions except $d=2$ and $d=4$. The proofs for the above dimensions differ by technical details only.

Let us construct a solution assuming that operator $\hat{q}$ in (24) is of the first order. We may write down solution of (24) in the following form,

$$P(\zeta)=\alpha G(\zeta,\zeta')+\beta\frac{\partial G(\zeta,\zeta')}{\partial\zeta_1'}+\Psi(\zeta;\zeta') \tag{25}$$

where α and β are ζ-independent parameters, $\Psi(\zeta;\zeta')$ is a homogenous solution, i.e. $\mathcal{L}[\Psi(\zeta;\zeta')]=0$, and G is a fundamental solution

$$\mathcal{L}[G(\zeta,\zeta')]=\sigma_{d-1}\delta(\zeta_1-\zeta_1')\ldots\delta(\zeta_d-\zeta_d'). \tag{26}$$

Let us also introduce fundamental solution of the Laplace operator in d-dimensions

$$\Delta[G_0(\zeta,\zeta')] = \sigma_{d-1}\delta(\zeta_1-\zeta_1')\dots\delta(\zeta_d-\zeta_d'), \quad G_0 = \rho^{2-d}, \quad \rho = \sqrt{(\zeta_1-\zeta_1')^2+\dots+(\zeta_d-\zeta_d')^2} \tag{27}$$

The fundamental solution G of $\mathcal{L}$ can be found applying Hadamard's expansion, following procedure developed for QAIS in [4, 5]. Here we briefly describe it:

Consider a QAIS with the Hamiltonian

$$H = \Delta + 2\Delta[\log\tau(\zeta)] = \xi(\zeta)\nabla\xi(\zeta)^{-2}\nabla\xi(\zeta)$$

and the corresponding free-boundary problem is governed by the elliptic operator

$$\mathcal{L} = \nabla\xi(\zeta)^{-2}\nabla, \tag{28}$$

where $\tau(\zeta)$ is a polynomial in $\zeta_1,\dots,\zeta_d$. According to [4, 5], the fundamental solution (26) can be written explicitly as follows

$$G(\zeta,\zeta') = \frac{\xi(\zeta)\xi(\zeta')}{\tau(\zeta')}\sum_{i=0}^{n}\mathcal{T}_i\Big[\Delta^{-i}[G_0(\zeta,\zeta')]\Big], \tag{29}$$

where $n = \deg(\tau(\zeta))$, and operators $\mathcal{T}_i$, acting on $\Delta^{-i}[G_0]$, are defined recursively as

$$\mathcal{T}_{i+1} = \mathcal{T}_i\Delta - H\mathcal{T}_i, \quad \mathcal{T}_0 = \tau(\zeta). \tag{30}$$

When $\mathcal{T}_i$ vanish for $i > n$, the Hadamard's expansion (29) truncates, which is a property of QAIS. Note that, as follows from the last equation, $\mathcal{T}_n$ is nothing but the intertwinning operator and one can also consider recursive procedure (30) as a test for algebraic integrability of a system with given $\tau(\zeta)$ (or $\xi(\zeta)$) accompanied by simultaneous construction of the intertwinning identity $H\mathcal{T}_n = \mathcal{T}_n\Delta$.

In our case $\tau(\zeta) = \xi(\zeta) = \zeta_1$, $n = 1$, and

$$\mathcal{T}_0 = \zeta_1, \quad \mathcal{T}_1 = 2\frac{\partial}{\partial\zeta_1} - \frac{2}{\zeta_1}.$$

Taking into account that in d dimensions (except $d = 2$ and $d = 4$)

$$\Delta^{-1}[G_0](\rho) = \frac{-\rho^2}{2(d-4)}G_0(\rho),$$

we get

$$G = \left(\zeta_1\zeta_1' - \frac{\rho^2}{d-4}\right)\rho^{2-d}. \tag{31}$$

Substituting

$$\alpha = 1, \quad \beta = \frac{r^2}{\zeta_1' d}, \quad \Psi(\zeta;\zeta') = -\frac{\rho^2 + 2\zeta_1\zeta_1' + (d-2)\zeta_1^2}{r^{2-d}d} + \frac{r^{4-d}}{d-4} \tag{32}$$

into (25), and taking (31) into account, we get

$$P(\zeta) = \left(\zeta_1\zeta_1' - \frac{\rho^2}{d-4}\right)\rho^{2-d} - \frac{r^2}{d}\left((d-2)\zeta_1(\zeta_1'-\zeta_1)\rho^{-d} - \rho^{2-d}\right) - \frac{\rho^2 + 2\zeta_1\zeta_1' + (d-2)\zeta_1^2}{r^{2-d}d} + \frac{r^{4-d}}{d-4}$$

It follows that P vanishes at the boundary $\rho = |\zeta - \zeta'| = r$ and therefore P is a solution of free-boundary problem for expanding sphere. From (25), (26) we get

$$\mathcal{L}[P](\zeta) = \sigma_{d-1}\left(1 - \frac{r^2}{\zeta_1' d}\frac{\partial}{\partial\zeta_1}\right)[\delta(\zeta_1 - \zeta_1')\ldots\delta(\zeta_d - \zeta_d')].$$

Applying procedure of derivation of quadrature identities, given in section 3, and taking the above equation into account, we complete the proof of the Proposition.

Similar propositions could be proved for other d-dimensional analogs of planar integrable elliptic growth systems mentioned in this paper.

In the planar case, rather than using conformal mappings, one could define the polynomial domains as the Laplacian quadrature domains formed by fluid injected through a multipole source into an initially empty homogeneous medium. Such a definition of the polynomial domains, as Laplacian domains, is acceptable in any dimension.

It is, however, unclear if higher-dimensional analogs of theorems considered in this paper are also valid for the $d > 2$ Laplacian quadrature domains other than balls.

In general, proofs use the following procedure: By Green's theorem

$$\int_\Omega \left(P(\zeta)\mathcal{L}[G](\zeta,\tilde\zeta) - G(\zeta,\tilde\zeta)\mathcal{L}[P](\zeta)\right)d^d\zeta = \int_{\partial\Omega}\left(P(\zeta)\xi(\zeta)^{-2}\partial_n G(\zeta,\tilde\zeta) - G(\zeta,\tilde\zeta)\xi(\zeta)^{-2}\partial_n P(\zeta)\right)ds$$

and by (2), (3), (24), (26) we have

$$P(\zeta) = \hat{q}[G](\zeta,\zeta') - \int_{\partial\Omega} G(\zeta,\tilde\zeta)\eta(\tilde\zeta)v_n(\tilde\zeta)d\tilde{s}.$$

Taking again into account the boundary condition (3), we get the equation for (constant) coefficients of $\hat{q}$:

$$\hat{q}[G](\zeta,\zeta') = \int_{\partial\Omega} G(\zeta,\tilde\zeta)\eta(\tilde\zeta)v_n(\tilde\zeta)d\tilde{s}, \quad \zeta,\tilde\zeta\in\partial\Omega. \tag{33}$$

To further simplify solution of this system one has to use (29) and properties of Laplacian domains (e.g. $v_n = dr(t)/dt$ in the case of the sphere).

6 Conclusion and open questions

In this article we have considered new applications of the theory of Quantum Algebraically Integrable Systems (QAIS) to the fluid dynamics of free boundary viscous flows. In two dimensions, we used conformal mapping techniques in combination with conservation laws in the form of quadrature identities to solve whole classes of inverse free-boundary problems. In higher-dimensions

we applied a "brute force" approach solving free-boundary problems directly. This approach relies on methods developed in the theory of QAIS, which limits us to a quite restricted class of problems. It will be interesting to find more general and perhaps less technical approaches for higher-dimensional cases.

It is worthy of mention related classification problems, namely the problem of description of all elliptic operators (or media, in the language of fluid dynamics) for which polynomial evolution of the free-boundary (9) can be sustained by a finite number of sources. This question seems to be connected with an open problem of classification of QAIS [3, 7, 6].

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